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General Dynamics of Two Distinct BPS Monopoles

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Classical and quantum dynamics of two distinct BPS monopoles in the case of non-aligned Higgs fields are studied on the basis of the recently determined low energy effective theory. Despite the presence of a specific potential together with a kinetic term provided by the metric of a Taub-NUT manifold, an O(4) or O(3,1) symmetry of the system allows for a group theoretical derivation of the bound-state spectrum and the scattering cross section.

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Recently, much attention has been given to the low energy interaction of BPS monopoles in the case that a semi-simple gauge group G of rank k breaks to its maximal torus $U(1)^k$ by one or more Higgs fields in the adjoint representation. In the simplest non-trivial case of the rank two and when the Higgs expectation values have only one independent component, the relative dynamics of two distinct fundamental BPS monopoles can be described by the kinetic Lagrangian provided by the metric of a Taub-NUT manifold [1, 2]. Here, relevant variables consist of the relative position $\mathbf{r}$ and the relative phase $\psi \in [0, 4\pi]$, and the Lagrangian takes the form

$$L_{\text{kin}} = \frac{\mu}{2}\left\{\left(1+\frac{r_0}{r}\right)\dot{\mathbf{r}}^2 + r_0^2\left(1+\frac{r_0}{r}\right)^{-1}(\dot{\psi} + \mathbf{w}(\mathbf{r})\cdot\dot{\mathbf{r}})^2\right\}, \tag{1}$$

where μ is the reduced mass, and $\mathbf{w}(\mathbf{r})$ is a Dirac monopole potential satisfying $\nabla \times \mathbf{w} = -\mathbf{r}/r^3$. The positive parameter r_0 of length dimension provides the length scale of the problem. The metric of the Taub-NUT space is chosen so that $L_{\text{kin}} = \mu g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu/2$ with the coordinates $x^\mu = (\mathbf{r}, r_0\psi)$. On the other hand, if the Higgs expectation values have more independent components, it has been shown recently [3] that the low energy Lagrangian acquires an additional attractive potential term. (For more than two distinct monopoles, there can be more than one term.) That is, the Lagrangian is of the form

$$L = L_{\text{kin}} - U(r) \tag{2}$$

with

$$U(r) = \frac{1}{2\mu r_0^2}\frac{a^2}{1+\frac{r_0}{r}}, \tag{3}$$

where $a > 0$ is a constant related to the degree of the vacuum misalignment.

In this paper we will present the analysis on classical and quantum dynamics of the system defined by the Lagrangian (2) and exhibit physical consequences due to the potential (3).

With $U(r) = 0$ and the parameter r_0 taken to be negative, our system reduces to the previously considered case in Ref. [4, 5, 6] for an approximate description of the relative dynamics concerning two identical BPS monopoles. In this case the system was found to possess the O(4)/O(3,1) dynamical symmetry analogous to the one in the well-known Coulomb problem, and this in turn allowed an algebraic derivation of the dyonic bound state spectrum and the two monopole scattering cross section as shown in Ref. [4, 5, 6]. This analysis requires no essential modification for our case, except that no dyonic bound states become available when the potential $U(r)$ is absent. In our case the relative charge leads to a repulsion instead of an attraction between two BPS monopoles.

What changes would the presence of the potential (3) in the case of non-aligned Higgs fields bring? What is remarkable about our system is that the O(4)/O(3,1) dynamical symmetry operates even in the presence of this potential and so we can determine the effect due to the potential exactly. Moreover, we will see that at the level of a time-independent Schrödinger equation, our system shares the same form as the Zwanziger system [7]-a model involving a suitably combined monopole+ Coulomb $+1/r^2$ potential with O(4)/O(3,1) symmetry. Based on this observation, we can deduce the exact spectrum of dyonic bound states (which exist only in the non-aligned Higgs vacuum), and also the scattering cross section for two distinct BPS monopoles in the general case.

We will study the classical dynamics first. Suppose that we have not yet posited the precise form of a radial potential $U(r)$ in Eq. (2). The Lagrangian has two obvious conserved quantities associated with the cyclic variable ψ and the time t, viz.,

$$q = \mu r_0^2 (1 + \frac{r_0}{r})^{-1} (\dot{\psi} + \mathbf{w}(\mathbf{r}) \cdot \dot{\mathbf{r}}), \tag{4}$$

$$E = \frac{\mu}{2}(1 + \frac{r_0}{r}) \left[\dot{\mathbf{r}}^2 + \frac{q^2}{\mu^2 r_0^2} \right] + U(r), \tag{5}$$

corresponding to the relative electric charge and energy, respectively. The mechanical three momentum

$$\boldsymbol{\pi} = \mu(1 + \frac{r_0}{r})\dot{\mathbf{r}} \tag{6}$$

satisfies the equation of motion

$$\dot{\boldsymbol{\pi}} = -\frac{\mu r_0}{2}\frac{\mathbf{r}}{r^3} \left(\dot{\mathbf{r}}^2 - \frac{q^2}{\mu^2 r_0^2} \right) - q\frac{\dot{\mathbf{r}} \times \mathbf{r}}{r^3} - \hat{\mathbf{r}}\, U'(r). \tag{7}$$

There is also the conserved angular momentum,

$$\mathbf{J} = \mathbf{r} \times \boldsymbol{\pi} + q\hat{\mathbf{r}}, \tag{8}$$

and so the position vector $\mathbf{r}$ moves on a cone defined by

$$\mathbf{J} \cdot \hat{\mathbf{r}} = q. \tag{9}$$

At this point, let us further demand, as a restriction on the potential $U(r)$, that all bound trajectories in the system be closed orbits. Such is the case if there is an additional conserved quantity of the Runge-Lenz type, $\mathbf{K} = \boldsymbol{\pi} \times \mathbf{J} - \beta\hat{\mathbf{r}}$ with a constant β. A simple calculation using Eq. (7)

determines the potential to be of the form (3) up to a possible constant shift. The precise form of the conserved Runge-Lenz vector is also fixed as

$$\mathbf{K} = \boldsymbol{\pi} \times \mathbf{J} - \mu r_0 \left(E - \frac{q^2}{\mu r_0^2} \right) \hat{\mathbf{r}}. \tag{10}$$

(Essentially, this corresponds to a generalization of the Bertrand theorem [8] to the case with the Taub-NUT kinetic term.) Our potential thus assumes a very special status-it corresponds to the general attractive potential which admits both the angular momentum and the Runge-Lenz vector as conserved quantities.

One may wonder whether there exists any additional modification of the Taub-NUT dynamics without jeopardizing the existence of a conserved Runge-Lenz type vector. One such modification [9] is to change the Taub-NUT metric to

$$ds^2 = \frac{a + br}{r} d\mathbf{r}^2 + \frac{ar + br^2}{1 + cr + dr^2} (d\psi + \mathbf{w}(\mathbf{r}) \cdot d\mathbf{r})^2. \tag{11}$$

For a particle moving in the space with this metric, there still exists a conserved Runge-Lenz vector [9]. However, the metric is always singular somewhere near the origin. This modification is rather ad hoc and does not have any good physics motivation. This metric also has nonzero Riemann tensor $R_{\mu\nu}$ and so is not hyperkähler. Further exploration of this modification was studied in Ref. [9, 10]. Another possibility, which is somewhat novel, is to add a term linear in velocity,

$$L_{\text{linear}} = \frac{c}{1 + \frac{r_0}{r}} (\dot{\psi} + \mathbf{w}(\mathbf{r}) \cdot \dot{\mathbf{r}}). \tag{12}$$

This term can be absorbed by shifting ψ to $\psi + ct/(\mu r_0^2)$ and $a^2 \to a^2 + c^2$ in the Lagrangian (2). Thus the physics does not change much due to this modification. Thus, we will focus on our Lagrangian (2).

Even in the presence of the potential (3), a complete description of the motion can now be given on the basis of the conservation laws. From two conserved quantities $\mathbf{J}$ and $\mathbf{K}$, we get the conserved quantities

$$\mathbf{J} \cdot \mathbf{K} = -\mu r_0 q (E - \frac{q^2}{\mu r_0^2}), \tag{13}$$

$$\mathbf{K}^2 = 2\mu (\mathbf{J}^2 - q^2)(E - \frac{q^2 + a^2}{2\mu r_0^2}) + \mu^2 r_0^2 (E - \frac{q^2}{\mu r_0^2})^2. \tag{14}$$

Defining a conserved vector

$$\mathbf{N} = q\mathbf{K} + \mu r_0 \left(E - \frac{q^2}{\mu r_0^2} \right) \mathbf{J}, \tag{15}$$

we also find that

$$\mathbf{N} \cdot \mathbf{r} = q(\mathbf{J}^2 - q^2). \tag{16}$$

From this equation, it follows that the motion is confined on a plane perpendicular to the constant vector $\mathbf{N}$. Thus the trajectories belong to a plane intersecting the cone, i.e., correspond to conic sections. The detailed orbit characteristics can be inferred from the energy equation (5). From Eq. (5), the effective potential for a given q is

$$U_{\rm eff}(r) = \frac{q^2}{2\mu r_0^2}(1 + \frac{r_0}{r}) + \frac{1}{2\mu r_0^2}\frac{a^2}{1+\frac{r_0}{r}}. \tag{17}$$

For $|q| \geq a$, $U_{\rm eff}(r)$ is a monotonically decreasing function of r and so there is no bound state. In this case, the energy is bounded by the minimum of the effective potential $U(\infty)$, i.e., $E \geq (q^2 + a^2)/(2\mu r_0^2)$. When the equality holds for this bound, the trajectories are parabolic orbits, and otherwise we find the hyperbolic orbits. On the other hand, if the relative electric charge satisfies $|q| < a$, the potential $U_{\rm eff}(r)$ assumes the minimum value $E_{\rm bps} = a|q|/(\mu r_0^2)$ at finite radius $r_{\rm bps} = r_0/(a/|q| - 1)$. Thus, with $|q| < a$, we have the following possibilities,

$E > \frac{1}{2\mu r_0^2}(q^2 + a^2)$: (unbounded) hyperbolic orbits,

$E = \frac{1}{2\mu r_0^2}(q^2 + a^2)$: parabolic orbit,

$\frac{1}{2\mu r_0^2}(q^2 + a^2) > E > E_{\rm bps}$: (bounded) elliptic orbits,

$E = E_{\rm bps}$: a static configuration with $\dot{\mathbf{r}} = 0$, $r = r_{\rm bps}$ and $\mathbf{J} = q\hat{\mathbf{r}}$.

The last case where $E = E_{\rm bps}$ can be regarded as the classical 1/4 BPS configuration [11, 3].

For the scattering orbits, the two conserved quantities $\mathbf{J}$ and $\mathbf{K}$ can also be utilized in determining how the scattering angle Θ depends on the impact parameter b, the initial relative speed v (or the initial energy E), and the relative charge q. Since the detailed arguments on this can be found from Ref. [4], we shall here give only the formula pertaining to our case:

$$\tan\frac{\Theta}{2} \;=\; \frac{1}{b}\sqrt{\frac{q^2}{\mu^2 v^2} + \frac{r_0^2}{\mu^2 v^4}\left(E - \frac{q^2}{\mu r_0^2}\right)^2} \tag{18}$$

$$= \frac{r_0}{2b}\sqrt{\left(1+\frac{a^2+q^2}{\mu^2 r_0^2 v^2}\right)^2 - \frac{4a^2q^2}{\mu^4 r_0^4 v^4}}\,, \tag{19}$$

where we have used the relation

$$E = \frac{1}{2}\mu v^2 + \frac{1}{2\mu r_0^2}(q^2+a^2) \tag{20}$$

for the initial variables. This leads to the Rutherford-type differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| \tag{21}$$

$$= \frac{r_0^2}{16}\left\{\left(1+\frac{a^2+q^2}{\mu^2 r_0^2 v^2}\right)^2 - \frac{4a^2q^2}{\mu^4 r_0^4 v^4}\right\}\csc^4\frac{\Theta}{2}\,. \tag{22}$$

Observe that when a vanishes, this reduces to the old result [4, 5, 6].

Let us now turn to quantum dynamics. For the classical Lagrangian (2), the canonical momentum conjugate to $\mathbf{r}$ is given by

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = \boldsymbol{\pi} + q\mathbf{w}(\mathbf{r}), \tag{23}$$

and the Hamiltonian by

$$H = \frac{1}{2\mu(1+r_0/r)}(\boldsymbol{\pi}^2 + \frac{a^2}{r_0^2}) + \frac{1}{2\mu}(1+\frac{r_0}{r})\frac{q^2}{r_0^2}. \tag{24}$$

In the quantum theory, dynamical variables become operators and so, for an expression involving operators like our Hamiltonian, an appropriate ordering must be specified. We will here follow the standard procedure developed for a general nonlinear system [12]. The variables $x^\mu = (\mathbf{r}, r_0\psi)$ and their conjugates $p_\mu = (\mathbf{p}, q/r_0)$ are taken to be hermitian operators with respect to the Taub-NUT volume measure $\sqrt{g}d^4x = (1+r_0/r)d^4x$ with $g = \det(g_{\mu\nu})$, and satisfy the basic commutation relations $[x^\mu, x^\nu] = 0, [p_\mu, p_\nu] = 0$ and $[x^\mu, p_\nu] = i\hbar\delta^\mu_\nu$. (From now on, we put $\hbar = 1$.) In this representation, the canonical momentum operator is

$$p_\mu = -i(\partial_\mu + \frac{1}{4}g^{-1}\partial_\mu g). \tag{25}$$

The Hamiltonian operator is then identified with [12]

$$H = \frac{1}{2\mu}g^{-1/4}p_\mu\sqrt{g}g^{\mu\nu}p_\nu g^{-1/4} + \frac{1}{2\mu r_0^2}\frac{a^2}{1+\frac{r_0}{r}} \tag{26}$$

$$= -\frac{1}{2\mu}\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu) + \frac{1}{2\mu r_0^2}\frac{a^2}{1+\frac{r_0}{r}}. \tag{27}$$

The relative charge operator $q = r_0 p_4 = -i\partial/\partial\psi$ commutes with the Hamiltonian and has eigenvalues s with $s = 0, \pm 1/2, \pm 1, \ldots.$ If one also introduces a non-hermitian operator

$$\boldsymbol{\pi} = -i\nabla - q\mathbf{w}(\mathbf{r}), \tag{28}$$

which satisfies the commutation relations

$$[\pi_i, x^j] = -i\delta_i^j, \quad [\boldsymbol{\pi}, \psi] = i\mathbf{w}, \quad [\pi_i, \pi_j] = -iq\epsilon_{ijk}\frac{x^k}{r^3}, \tag{29}$$

it is a matter of lengthy computations [6] to show that the Hamiltonian (27) can be recast precisely into the form (24), now with the operator ordering taken into consideration. The Hamiltonian operator in Eq. (24) is hermitian with respect to the Taub-NUT volume measure.

We are interested in the solution of the time-independent Schrödinger equation, $H\Psi = E\Psi$. For the eigenstates with $q = s$, we use the Hamiltonian (24) and factor out the trivial phase $e^{iq\psi}$ from the wave function Ψ to obtain

$$\left\{\frac{1}{2\mu}\boldsymbol{\pi}^2 - \frac{\alpha}{r} + \frac{q^2}{2\mu r^2} - \mathcal{E}\right\}\Psi = 0, \tag{30}$$

where two new constants α and $\mathcal{E}$ are given by

$$\alpha = r_0(E - \frac{q^2}{\mu r_0^2}), \quad \mathcal{E} = E - \frac{1}{2\mu r_0^2}(a^2 + q^2). \tag{31}$$

Disregarding the above relations, one may look upon Eq. (30) as the Schrödinger equation of a flat-space system, $\mathcal{H}\Psi = \mathcal{E}\Psi$, with the Hamiltonian,

$$\mathcal{H} = \frac{1}{2\mu}\boldsymbol{\pi}^2 - \frac{\alpha}{r} + \frac{q^2}{2\mu r^2}. \tag{32}$$

This Hamiltonian is hermitian with respect to the usual Euclidean three space volume measure and contains, aside from the magnetic monopole and Coulomb-type potentials, a $1/r^2$ repulsive potential of a definite strength. The very system was first considered by Zwanziger [7]; Eq. (32) describes the relative dynamics in his exactly soluble nonrelativistic model of two dyons. It enjoys the $O(4)/O(3,1)$ symmetry. Our system also enjoys the $O(4)/O(3,1)$ symmetry, which was noticed earlier without the potential (3) [5, 6]. In our case the symmetry generators are quantum operators corresponding to the angular momentum and the Runge-Lenz vector:

$$\mathbf{J} = \mathbf{r} \times \boldsymbol{\pi} + q\hat{\mathbf{r}}, \tag{33}$$

$$\mathbf{K} = \frac{1}{2}(\boldsymbol{\pi} \times \mathbf{J} - \mathbf{J} \times \boldsymbol{\pi}) - \mu r_0 \hat{\mathbf{r}}(H - \frac{q^2}{\mu r_0^2}), \tag{34}$$

where H is our full Hamiltonian operator (24). (Note that the order of various operators is important now.) Then by a bit of lengthy algebra, one can verify that they satisfy the following relations:

$$[\mathbf{J}, H] = [\mathbf{K}, H] = 0, \tag{35}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \tag{36}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k, \tag{37}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k (2\mu H - \frac{q^2 + a^2}{r_0^2}). \tag{38}$$

They also satisfy

$$\mathbf{K} \cdot \mathbf{J} = \mathbf{J} \cdot \mathbf{K} = -\mu r_0 q (H - \frac{q^2}{\mu r_0^2}), \tag{39}$$

$$\mathbf{K}^2 - \mu^2 r_0^2 (H - \frac{q^2}{\mu r_0^2})^2 = 2\mu(\mathbf{J}^2 - q^2 + \hbar^2)(H - \frac{q^2 + a^2}{2\mu r_0^2}). \tag{40}$$

With the explicit $\hbar$ correction in the quantum expression term, one can compare these equations with their classical counterparts (13) and (14). For the Zwanziger system, the corresponding $O(4)/O(3,1)$ generators are made of $\mathbf{J}$ and

$$\tilde{\mathbf{K}} = \frac{1}{2}(\boldsymbol{\pi} \times \mathbf{J} - \mathbf{J} \times \boldsymbol{\pi}) - \mu\alpha\hat{\mathbf{r}}, \tag{41}$$

which is obtained from $\mathbf{K}$ by replacing the operator $r_0(H - q^2/\mu r_0^2)$ with the parameter α of Eq. (31). They commute with $\mathcal{H}$ and satisfy the similar algebra with $[\tilde{K}_i, \tilde{K}_j] = -i\epsilon_{ijk} J_k (2\mu\mathcal{H})$, and so on. The analogue of Eqs. (39) and (40) for these operators are

$$\tilde{\mathbf{K}} \cdot \mathbf{J} = \mathbf{J} \cdot \tilde{\mathbf{K}} = -\mu\alpha q \tag{42}$$

$$\tilde{\mathbf{K}}^2 - \mu^2\alpha^2 = 2\mu(\mathbf{J}^2 - q^2 + \hbar^2)\mathcal{H} \tag{43}$$

As in the Coulomb system [13], the bound state spectrum and the scattering cross section for the Zwanziger system can be found exactly with the help of an appropriate group representation theory. Then, they may be recast using the connection (31), to obtain the corresponding results for our original system with the Hamiltonian (24). Of course, it is possible to use the group representation theory directly with our system also. But the procedure just mentioned is simpler.

For $\alpha > 0$, the Zwanziger system allows bound states with negative energy $\mathcal{E} < 0$. In this case, the operator $\mathbf{M} = \tilde{\mathbf{K}}/\sqrt{2\mu|\mathcal{E}|}$ and $\mathbf{J}$ satisfy an $O(4)$ algebra. Introduce $\mathbf{A} = (\mathbf{J} + \mathbf{M})/2$ and

$\mathbf{B} = (\mathbf{J} - \mathbf{M})/2$ as the generators for two $O(3)'s$ of $O(4) = O(3) \times O(3)$. Then an energy eigenstate Ψ may be taken as $|a, a_3, b, b_3>$ with integers or half-integers for a, a_3 and b, b_3, satisfying

$$\mathbf{A}^2\Psi = a(a+1)\Psi, \quad A_3\Psi = a_3\Psi, \tag{44}$$

$$\mathbf{B}^2\Psi = b(b+1)\Psi, \quad B_3\Psi = b_3\Psi. \tag{45}$$

Then, making a judicious use of the identities (42) and (43), it is not difficult [7] to derive the exact bound state spectrum $\mathcal{E} = -\frac{\mu\alpha^2}{2n^2}$, where n is the principal quantum number restricted to take values $n = |s|+1, |s|+2, ...$ if the eigenvalue s of the operator q assumes half-integers or integers. This bound state energy levels are highly degenerate, since, for given s and n, the quantum number j associated with $\mathbf{J}^2 = (\mathbf{A}+\mathbf{B})^2$ can run from $|s|$ to $n-1$. We can now translate these findings into those for our original system. First of all, the conditions to allow bound states, $\alpha > 0$ and $\mathcal{E} < 0$, translate into

$$\frac{s^2}{\mu r_0^2} < E < \frac{1}{2\mu r_0^2}(a^2 + s^2), \tag{46}$$

and so our system can have bound states only with $|s| < a$ as in the classical case. On the other hand, inserting the formula $\mathcal{E} = -\frac{\mu\alpha^2}{2n^2}$ to Eq. (31), we obtain a quadratic equation for E,

$$-\frac{\mu r_0^2}{2n^2}(E - \frac{s^2}{\mu r_0^2})^2 = E - \frac{1}{2\mu r_0^2}(a^2 + s^2). \tag{47}$$

Solving this equation, we immediately obtain the desired bound state spectrum for our system:

$$E = \frac{1}{\mu r_0^2}\left\{-(n^2 - s^2) + n\sqrt{n^2 - s^2 + a^2}\right\} \tag{48}$$

with $|s| < a$ and $n \geq |s| + 1$. The degeneracy of each energy eigenstates is $n^2 - s^2$ as j runs from $|s|$ to $n-1$.

The case with $\mathcal{E} > 0$ corresponds to a scattering state. We may put $\mathcal{E} = \mu v^2/2$ with the initial speed v. The operators $\mathbf{M}$ and $\mathbf{J}$ now satisfy the O(3,1) algebra. As in Ref. [7], the group theory can be utilized to derive the exact quantum mechanical scattering cross section formula for the Zwanziger system,

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\mu^2 v^4}(s^2 v^2 + \alpha^2)\csc^4\frac{\Theta}{2}. \tag{49}$$

Again we translate this result into our case by using the relation (31) and $q = s$ and see that this coincides with the classical cross section formula (22), making the classical result to be quantum mechanically exact.

For a given value of s, the bound state energy spectrum in Eq. (48) is an increasing function of n, satisfying the inequality

$$E(n = |s| + 1) \leq E < E(n = \infty) = \frac{1}{2\mu r_0^2}(a^2 + s^2). \tag{50}$$

Especially, the ground state energy $E(n = |s| + 1)$ with degeneracy $2|s| + 1$ is greater than the classical BPS energy $E_{bps} = a|s|/(\mu r_0^2)$, which is identical to $E(n = |s|)$ if n could take the value $|s|$. The difference between the ground state energy and the classical minimum is due to the zero point energy of the bosonic theory. It would be interesting to formulate the most general supersymmetric Lagrangian [3, 14, 15] for our system and find BPS and non-BPS bound states. There would be no such energy difference in supersymmetric vacua.

Finally, there exists an even larger conformal symmetry group $O(4,2)/O(5,1)$ for the Taub-NUT system [5, 6]. This can be easily generalized to our system, which we do not discuss here. They will again play a role in the supersymmetric extension of our work.

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